PHYSICAL MODEL OF A 3-CONSTANT ISOTROPIC ELASTIC MATERIAL

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A study is presented of the statical deformation of a model of a supposed 3-constant isotropic elastic material which exhibits the phenomenon of couple stresses. A description of the design and construction of the physical model is given. The related theory is briefly summarized and used in order to determine the three independent elastic constants which characterize the model of the material. The specific constitutive equations are presented. Experimental results are given for the model loaded in a mode which may be called pure bending.

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INTRODUCTION

At the end of the nineteenth century L. Boltzmann questioned the universal validity of the principle of the symmetry of the stress tensor as related to an elastic continuum [1][3]. Also, about that time Kelvin and Voigt for various reasons considered the same question. As is now well-known, the Cosserat brothers then actually developed a theory of elasticity generalized to include the notion of couple stresses and the corresponding asymmetry of the stress tensor [2]. Although these latter two were civil engineers or maybe because of the fact they did not exploit the idea apparently considering that there were no practical applications. In any event, the fact is that there was a considerable lapse of interest until the last decade or so when a rather strong resurgence occurred. One reason for this is the rapidly growing knowledge of solid state physics. For example it was inevitable that interest in dislocation theory would lead to a reinvestigation of the sufficiency of the classical theory of elasticity. Evidence of such a trend is exemplified by the investigations of Kröner who introduced the concept of couple stress in his studies of the deformation of metals [3].

Apparently the notable elastician, R. D. Mindlin, also saw the need for greater generality in the formulation of the

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[3] Numbers in brackets designate references at end of paper.

theory of elasticity for he has published several papers on the subject [4, 5, 6]. His formulations are well presented for the purpose of applications, but so far he has not adduced definitive physical cases which demonstrate the existence of couple stress effects in real materials. In addition to the fairly extensive studies by Mindlin, there have been other theoretical investigations by Truesdell and Ericksen [7] as well as by Toupin [8]. In 1962 Schaefer published a paper [9] containing an analysis of deformation related to couple stresses. As noted by him, the delicate question in the theory arises in the introduction of the constitutive equations. In his analysis he assumes a particular type which is more or less convincing. One of the present writers was influenced in his thinking on the matter mainly by questions that may arise in connection with the nature of general strain fields. Specifically he was interested in the study of strain made by Leroux in 1911 [10]. It now seems somewhat odd that the older elasticians concentrated their thinking on the concepts of lineal strain and shear strain. When they finally introduced the connection of the strains to the stresses the constitutive equations naturally turned out to be the generalized Hooke's law which represented a linear relation between normal and shearing stress with lineal and shearing strain respectively. Leroux concentrated his attention on the gradients of the displacement field in a much more general manner than was the custom in his time.

In the present paper consideration is given to the implication of linear constitutive equations which postulate the existence of couple stresses related to the change of curvature of line elements, caused by straining an isotropic elastic solid. For simplicity and clarity in the development, the study is restricted to the case of plane strain. Furthermore, rather than seek out at the present time material which may show effects of couple stresses, it was considered more instructive to devise a mechanical model of a material to which the theory may apply. The specific method and results are presented in the paper.

EQUATIONS OF EQUILIBRIUM

Using the notation of Timoshenko, the 3-dimensional equations of equilibrium are those given by the Cosserat brothers [2]. They may be written:

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} = 0$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} = 0$$

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} = 0 \tag{1}$$

$$\frac{\partial q_{xx}}{\partial x} + \frac{\partial q_{yx}}{\partial y} + \frac{\partial q_{zx}}{\partial z} + \tau_{yz} - \tau_{zy} = 0$$

$$\frac{\partial q_{xy}}{\partial x} + \frac{\partial q_{yy}}{\partial y} + \frac{\partial q_{zy}}{\partial z} + \tau_{zx} - \tau_{xz} = 0$$

and

$$\frac{\partial q_{xz}}{\partial x} + \frac{\partial q_{yz}}{\partial y} + \frac{\partial q_{zz}}{\partial z} + \tau_{xy} - \tau_{yx} = 0$$

The first three equations are those usually given for the forces at a point in a continuum. The last three equations are those for moments at a point and they arise because of the postulated existence of couples at the point. It is this assumption, of course, which introduces the non-symmetric nature of the shearing stresses τ_{ij} .

The present investigation of the meaningfulness of the theory in terms of a physical model is limited to the two-dimensional case. Therefore the complete set of field equations defining the problem will be limited accordingly. Hence the equations of equilibrium may be written simply as:

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} = 0$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0 \qquad (2)$$

$$\frac{\partial q_{xz}}{\partial x} + \frac{\partial q_{yz}}{\partial y} + \tau_{xy} - \tau_{yx} = 0$$

STRAINS, CURVATURES, AND COMPATIBILITY

The strains and curvatures in terms of the displacements u, v may be written:

$$\epsilon_{xx} = \frac{\partial u}{\partial x}, \quad \epsilon_{yy} = \frac{\partial v}{\partial y}, \quad \epsilon_{xy} = \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)$$

and (3)

$$k_{xz} = \frac{\partial \omega_z}{\partial x}, \quad k_{yz} = \frac{\partial \omega_z}{\partial y}$$

ω_z is the rotation about the z-axis which is here assumed to be the axis parallel to the length of the long prism usually

assumed in plane strain theory.

In the classical plane strain theory there is only one compatibility equation which is not satisfied identically. In the present case the situation is somewhat more complicated and results in the following equations of condition which are readily verifiable.

$$\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = 2\,\frac{\partial^2 \varepsilon_{xy}}{\partial x\,\partial y}$$

$$k_{xz} = \frac{\partial \varepsilon_{xy}}{\partial x} - \frac{\partial \varepsilon_{xx}}{\partial y}$$

and (4)

$$k_{yz} = \frac{\partial \varepsilon_{yy}}{\partial x} - \frac{\partial \varepsilon_{xy}}{\partial y}$$

Note: $\dfrac{\partial k_{xz}}{\partial y} = \dfrac{\partial k_{yz}}{\partial x}$

CONSTITUTIVE EQUATIONS

The material substance studied in the present investigation is assumed to have a somewhat more complicated set of constitutive equations than the customary Hooke's law. They are written as follows:

$$\varepsilon_{xx} = \frac{(1 + \nu)}{E} [(1 - \nu)\sigma_x - \nu\sigma_y]$$

$$\varepsilon_{yy} = \frac{(1 + \nu)}{E} [-\nu\sigma_x + (1 - \nu)\sigma_y]$$

$$\varepsilon_{xy} = \frac{1}{2G} (\tau_{xy} + \tau_{yx})$$

and (5)

$$k_{xz} = \frac{1}{2C} q_{xz}$$

$$k_{yz} = \frac{1}{2C} q_{yz}$$

The first three relations are the usual Hooke's law written now to allow for the non-equality of τ_{xy} and τ_{yx}. The remaining two relations relate the couple stresses q_{xz}, q_{yz} to the respective changes of curvature k_{xz}, k_{yz}.

The three independent constants of elasticity are E, G, and C. The usual relation of classical elasticity holds for E, G, ν as follows:

$$G = \frac{E}{1 + \nu} \tag{6}$$

Note that G is written twice as large as usual. The ν is the usual Poisson's ratio.

PURE BENDING

Following the lead of Schaefer [9], the loading in Fig. 1 designated by (b) and (c) defines a case which may be appropriately referred to as "pure bending". Note that it differs from pure bending in classical elasticity by the addition of uniformly distributed point couples. As a consequence it turns out that all sections parallel to the loaded ends are stressed in the same manner. If one applies only the loading given by Fig. 1 (b) the sections parallel to the loaded ends will be stressed differently from those ends and probably in a complicated manner.

Using the theory of pure bending as now defined, it will be relatively easy to solve the equations of equilibrium and determine the displacements in terms of the elastic constants E, G, and C. For convenience in performing the experiments, the actual loading was obtained by simply adding a uniform tensile stress at the ends as shown in Fig. 1 (a). The solution for the equations of equilibrium may be easily verified to be:

$$q_{xz} = q = \text{constant}$$

$$q_{yz} = 0$$

$$\sigma_x = \frac{q}{2C}\frac{E}{1-\nu^2}(a - y) \tag{7}$$

$$\sigma_y = 0$$

and

$$\tau_{xy} = \tau_{yx} = 0$$

Integrating the strain equations in terms of the displacements u, v (3), the displacements are:

$$u = -\frac{q}{2C} yx + \frac{qa}{2C} x$$

$$v = \frac{\nu}{1-\nu}\frac{q}{4C} y^2 + \frac{q}{4C} x^2 - \frac{\nu}{2(1-\nu)}\frac{qa}{C} y \tag{8}$$

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = \frac{q}{2C} x$$

It should be noted that the solution requires that the boundary loads must be applied so that the magnitude of the distributed couple stress is in constant ratio to the normal stress σ_x at any given point.

Now if the displacements are measured on an experimental model loaded in the manner here assumed, the elastic constants can be calculated from Eq. (8).

EXPERIMENTAL MODEL

A mechanical model was devised as an analogue to a 3-constant isotropic elastic solid. In order to emphasize the influence of the couple stresses the model was fabricated out of what may be called "molecular" or cell-like sub-units. For the purpose, rigid solid blocks of aluminum fastened together with thin flat strips of steel were used. Each cellular unit then is composed

of four cubes fastened by the steel strips. Alternately, each unit might be thought of as a single cube with four strips extending from four of its faces. A sketch of the model is shown in Fig. 2 and a photograph of the model fastened in place in the loading apparatus is shown in Fig. 3. The steel strips with cross section 0.5" by 0.016" were cemented to the one inch aluminum cubes with epoxy resin. Obviously the model is not a point-wise homogeneous continuum as assumed in the theory. However, if the overall size is large compared to the dimensions of the cellular unit, the model may statistically approximate a homogeneous continuous material. Also, since the flat steel binding strips are all of the same dimensions and since the displacements occur only in planes parallel to the center plane, the model was considered to be approximately isotropic. In order to construct a three-dimensional model which is approximately isotropic, the flat steel strips could be replaced by round rods which extend one each from each of the six faces of a cube.

EXPERIMENTAL APPARATUS FOR DETERMINATION OF ELASTIC CONSTANTS

In order to deform the model as required by the previously developed theory of pure bending, a special apparatus was developed. It was constructed so that the required tensile loads and couples might be readily applied at two opposite sides of the model as shown in Fig. 3.

As can be seen, the model is placed in a horizontal position and rests lightly on a number of ball bearings located under the cubes in order to support its weight. A small round drill rod also supports the model along its axis of symmetry with respect to the loaded sides.

The loading device consists simply of dead weights which hang down from light steel wires which pass over small frictionless pulleys and then attach to the model at predetermined points. The wires which apply the tensile loads to the model are fashioned in a smooth loop arrangement which reduces undesired constraints on the boundary. They each carried initial loads of three pounds.

The couples were applied at each cube in a given boundary by means of parallel wires which pull in opposite directions. Special fittings used for the purpose are shown in Fig. 4. By means of these fittings the magnitude of the couples could be varied either by changing the moment arm or by changing the intensity of the force or by both.

The displacement measurements essential for the determination of the elastic constants were made with the aid of specially adapted gages. The gages consisted of aluminum foil strips 0.03" thick by 0.5" wide stretched positively between two predetermined points on the model. The elongation of the strips and hence the relative displacements between the two end points were determined with aid of electrical resistance strain gages

cemented to the foil. Standard circuitry was used for the measurements. In every case, two such foils were placed one on each side of the model and parallel to the model. The average of these two gages provided a reliable measure of the required displacement.

DETERMINATION OF ELASTIC CONSTANTS

With the experimental apparatus which was developed, two types of experiments were performed in order to determine the three elastic constants or moduli. The constants are the usual Young's modulus E, the shear modulus G, and the curvature change type modulus C. The latter modulus, of course, is the one related to the couple stresses.

The modulus E is obtained by means of the familiar tension experiment and no more need be mentioned here about it. The Poisson type constant ν was very close to zero for the particular model used. The results of this experiment may be briefly summarized as follows:

$$\nu \doteq 0$$

$$E' h \times 10^{-6} = 0.182 \text{ lb./in.}$$

where

$$E' = \frac{E}{1 - \nu^2}$$

$$G = \frac{E}{1 + \nu}$$

as previously mentioned.

The method of determination of the constant C is novel and so will be described somewhat at length.

Eq. (8) shows that if the displacement u and couple stress q are known at a point on the boundary, then C can be calculated. However, it is clear that Eq. (7) must be simultaneously satisfied; that is for the same q and the applied σ_x at the point the same value of C must be obtained. E' was obtained in the tension experiment and its value is known. Consequently, Eq. (7) and Eq. (8) must be solved simultaneously. This can be done graphically as shown in Fig. 5. The qh is plotted as function of displacement of a point A for a given applied stress σ_D (σ_D is the magnitude of the linearly distributed boundary tension at a point D). For example, the straight line given for the parameter $\sigma_D h$ equal to 16.85 pounds per inch is obtained by first loading the model with a stress distribution σ_x whose value at point D is σ_D . Then keeping the σ_x stresses fixed, the couples qh were increased gradually from zero and the corresponding changes in displacements at A were measured so that data for the displacement curve was obtained.

Now the displacement u at a point such as A , called u^*, can be calculated from Eqs. (7) and (8) by dividing σ_x by u and thereby eliminating q/C . The value of σ_x applied at A , E' and the value of x are then substituted in the resulting equation for u . We may erect a vertical straight line in

Fig. 5 at the abscissa given by the value u^* which was computed. The intersection of this line with the oblique line for which the parameter $\sigma_D h$ is equal 16.85 pounds per inch determines the required value of qh . With these data determined, either Eq. (7) or Eq. (8) can be used to calculate the value of C .

The same procedure may be followed for different values of $\sigma_D h$. In the present investigation this was done for four distinct values and the corresponding lines plotted. As a consequence, four parallel lines are plotted in Fig. 5 and for each one a particular point on it is determined by its coordinates u^*, qh . All four of these points so determined should lie on a straight line through the origin as shown.

Instead of measuring the displacement u^* at point A it can obviously be measured at any other point such as B , for example. In such a case the value for σ_x used in Eq. (7) must be that which is applied at point B . This was done for three points on the boundary as a further check of the method and the results are shown in Fig. 6 as three sets of lines.

Finally the correctness of the experimental results may be further investigated by examining the distribution of the displacements along the loaded boundaries for any given loading. Such was done and the results are plotted in Fig. 7. It is seen that the variation of displacement is practically linear as required by the theory.

DISCUSSION

It would seem that experimental models of the type studied in the present investigation serve the purpose of demonstrating the rôle and nature of couple stresses as they conceivably exist in fully continuous materials. Furthermore, since the experimental study required for the determination of the 3 constants contains some novel features it appears reasonable to assert that the present study could serve as a starting point for further and more elaborate investigation of the existence of couple stress effects in materials. While the rôle of couple stresses in the deformation of materials may be entirely second order as in the case of most steels say for example, it is quite possible that the effects could be primary for many other kinds of materials especially for certain types of loading. In any event, one cannot logically escape the present requirement to analyze much more closely the acceptability of the classical elasticity for the precise determination of deformation of any real material.

Finally, it may be said that it is fully realized that only one type of static experiment has been performed in the present case in order to calculate the values of the alleged 3 constants of isotropic elasticity. Further experiments with different kinds of loadings are certainly required. In fact, one of the present authors plans to perform dynamic experiments with models of the type used here and see if mode shapes and frequencies of vibration can be predicted properly.

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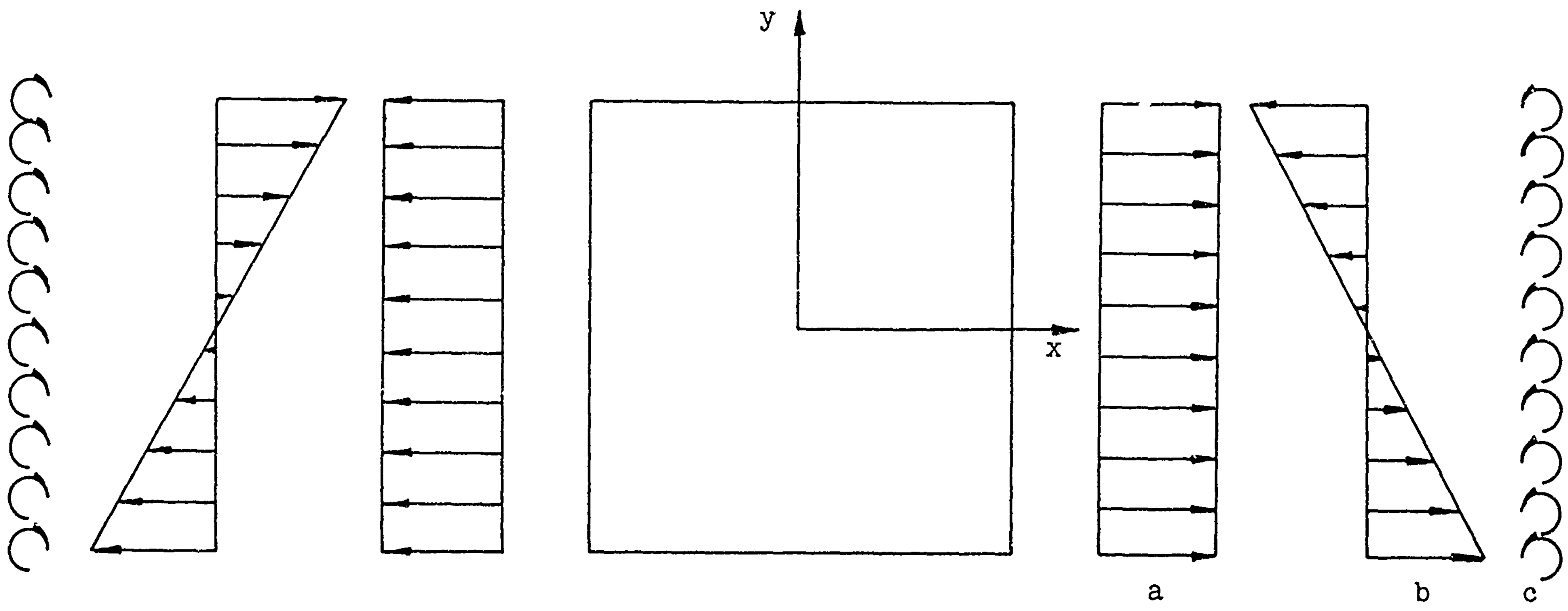

a - Uniform tensile stress
b - Linear varying stress
c - Uniform couple stress

Fig. 1 Loading on the model

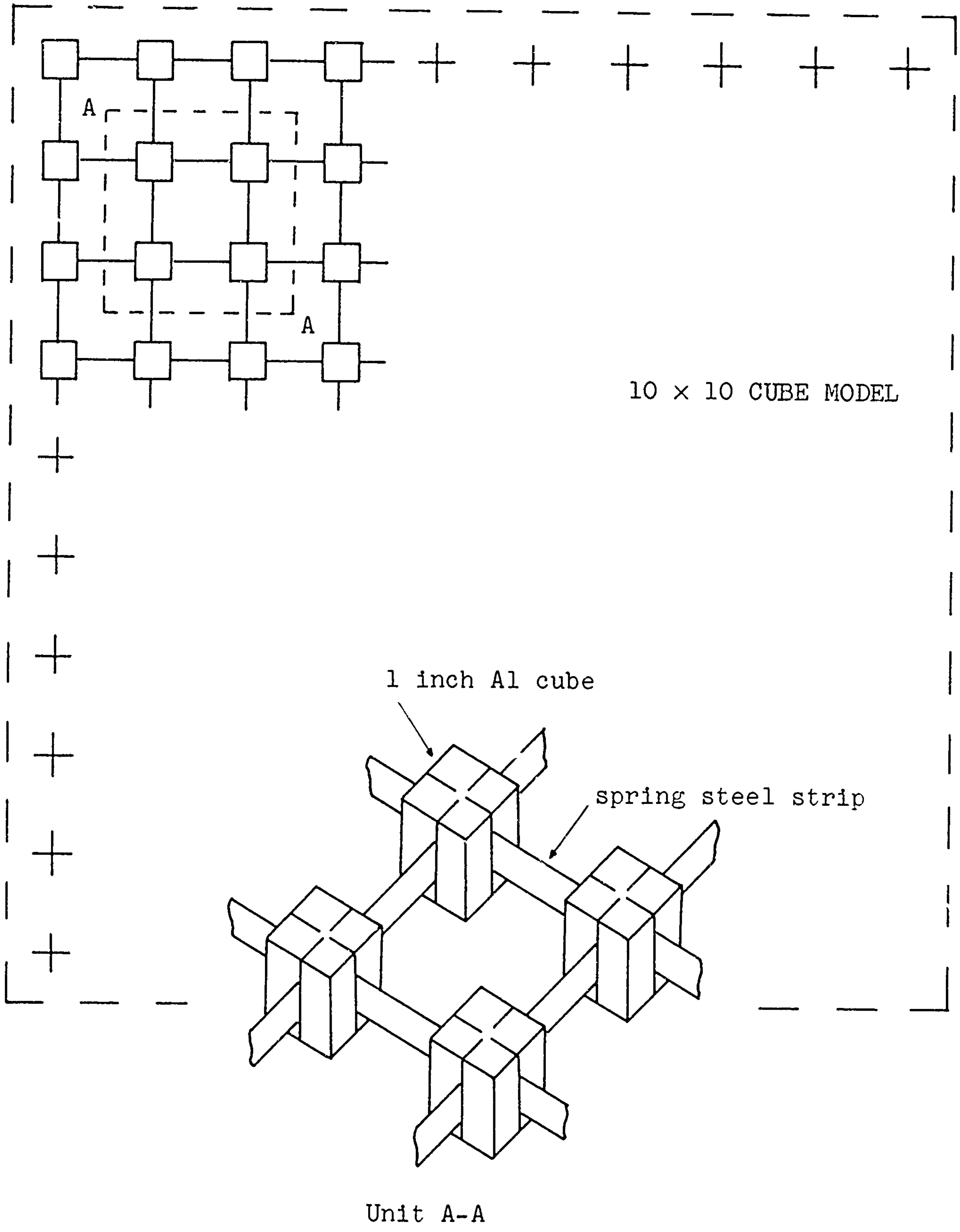

Fig. 2 Schematic diagram of model of 3-constant isotropic elastic material

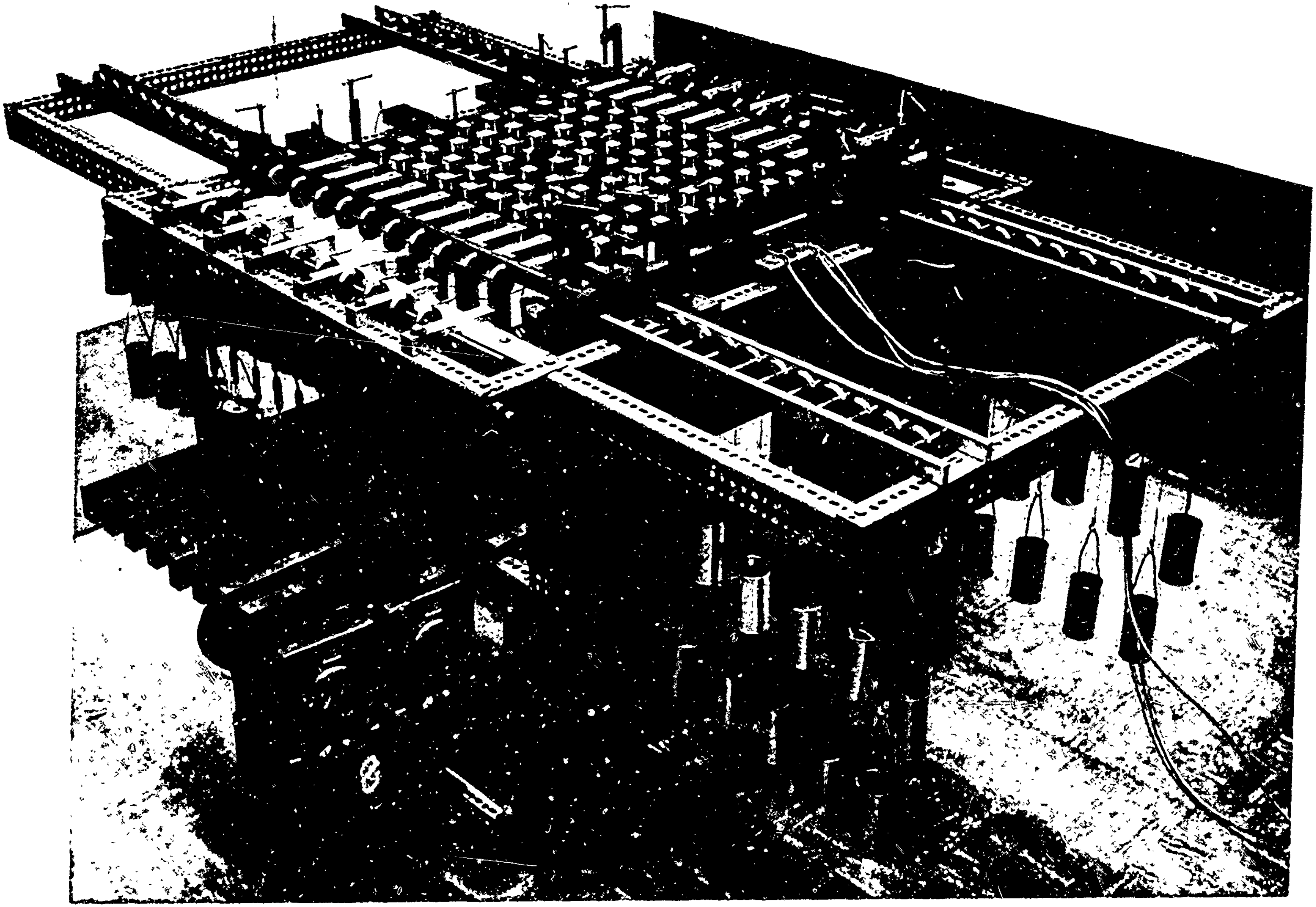

Fig. 3 Apparatus for determination of elastic constants

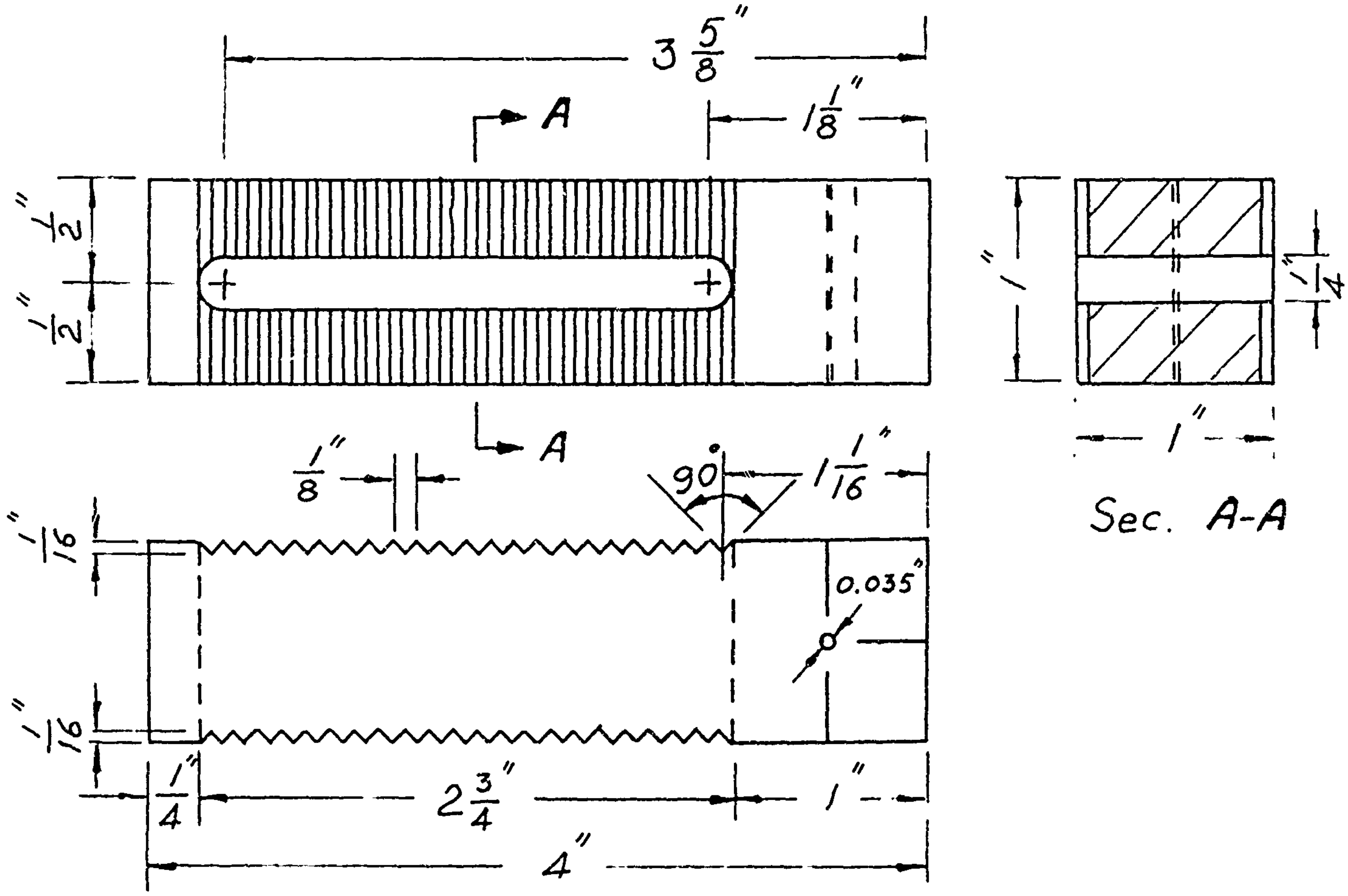

Fig. 4 Couple-applying device

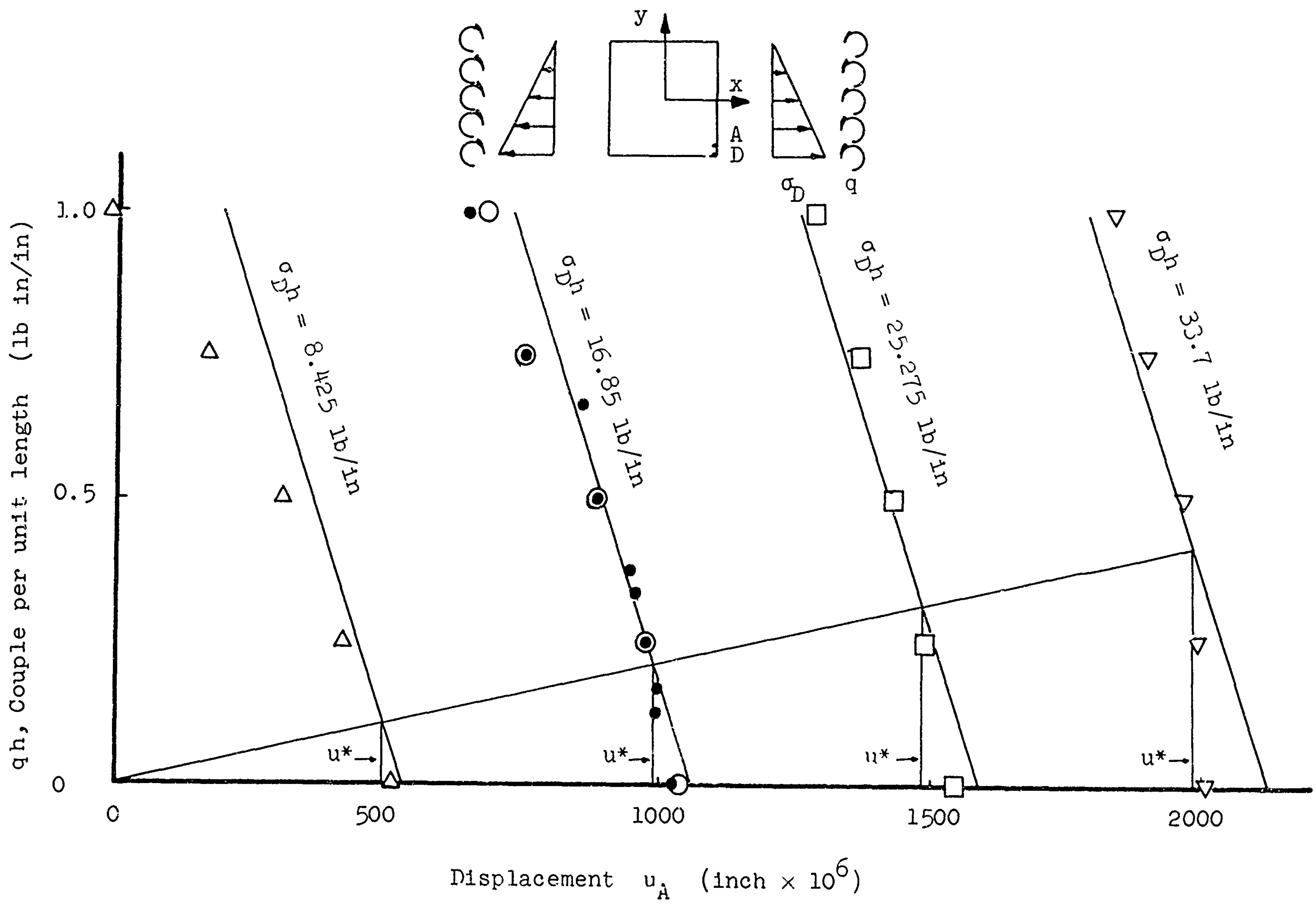

Fig. 5 Graphical construction for determination of elastic constant C.

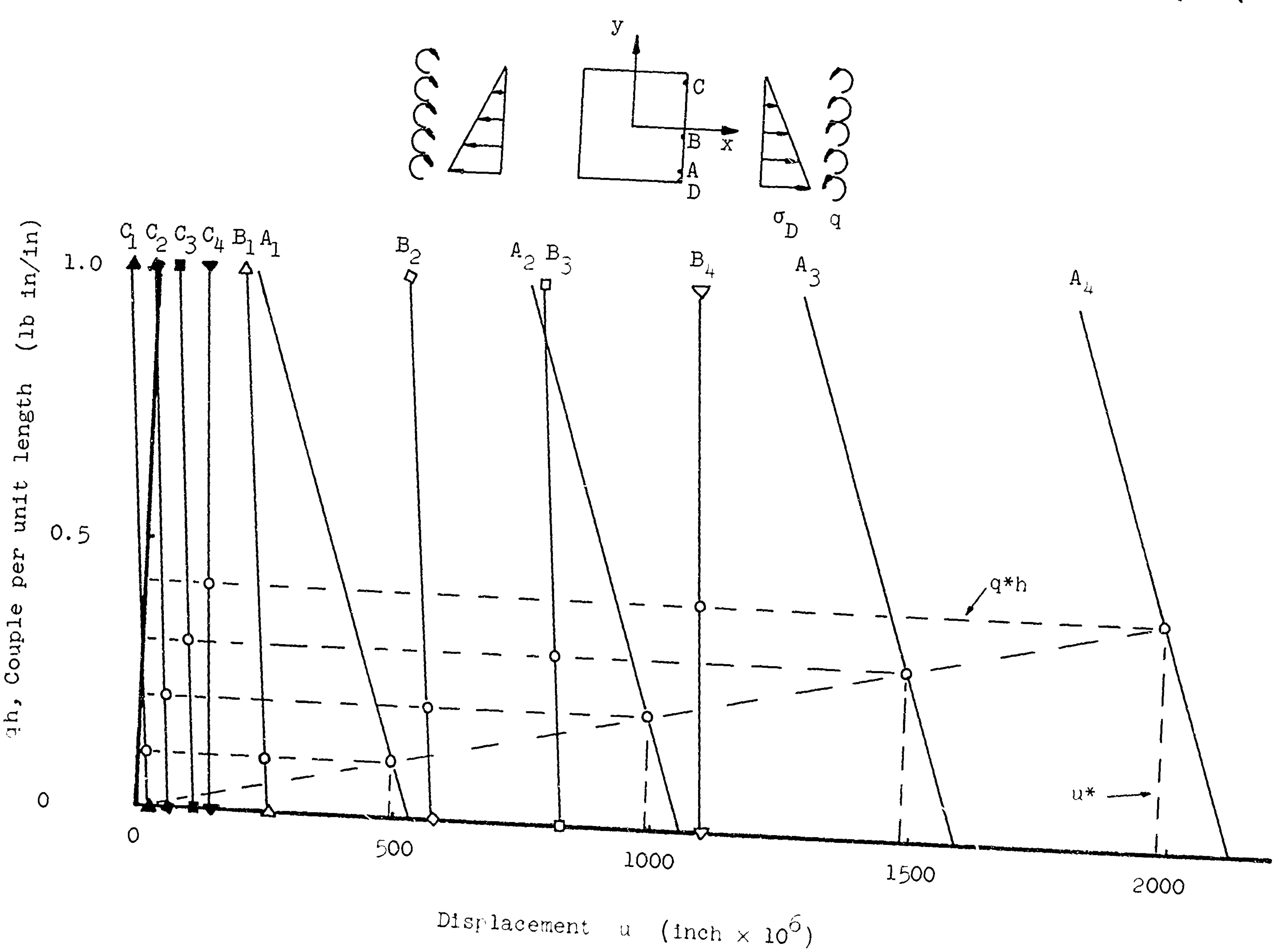

Fig. 6 Graphical constructions for elastic constant C using displacements at three points A, B and C.

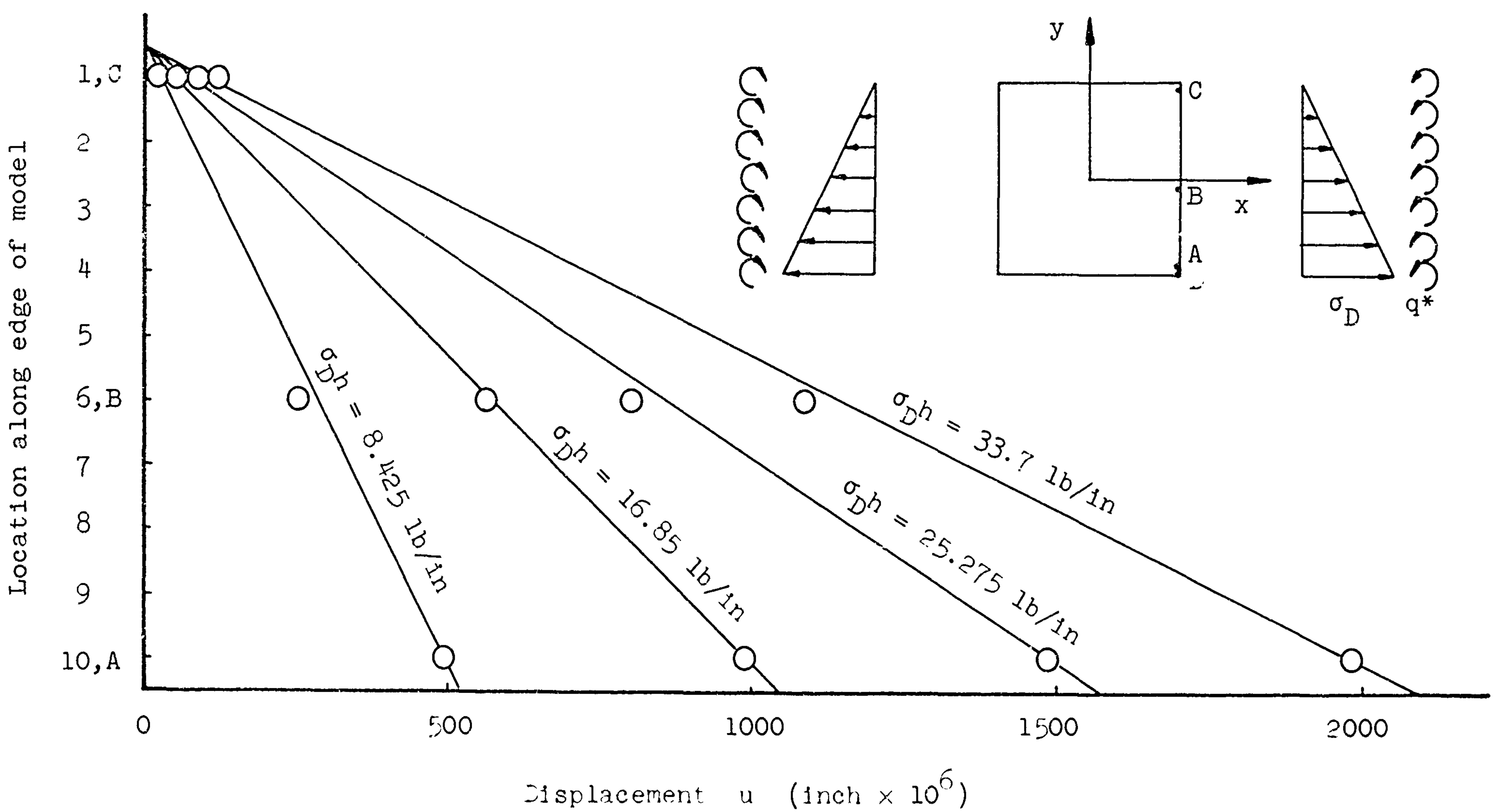

Fig. 7 Displacement u along loaded edge of model